Solitary Waves Propagating Along a Magnetic Field in a Warm Collisionless Plasma

Bhimsen K. Shivamoggi

Department of Theoretical Physics, Research School of Physical Sciences, The Australian National University, P. O. Box 4, Canberra, A.C.T., 2600 Australia

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A study is made of solitary waves propagating along a constant, uniform magnetic field in a warm collisionless plasma. The calculation is based on a two-fluid model of the plasma. It is found that the effect of the thermal motion of ions and electrons is to reduce the amplitude but enhance the velocity of propagation of the solitary wave.

1. Introduction

Adlam and Allen [1], and Davis et al. [2] have studied a class of large-amplitude one-dimensional hydromagnetic waves which propagate undistorted in a cold, collisionless plasma in a direction perpendicular to a constant, uniform magnetic field, and Montgomery [3] and Saffman [4] studied the waves propagating parallel to the magnetic field. Whereas Montgomery considered an infinite train of periodic waves, Adlam and Allen, Davis et al. and Saffman considered solitary waves. The purpose of this paper is to study solitary waves propagating along the magnetic field in a warm plasma. The propagation characteristics of such solitary waves are found to be influenced considerably by the thermal motion of the ions and the electrons.

2. Governing Equations

Consider a solitary wave in a warm, collisionless plasma propagating steadily with velocity U along a magnetic field. The conditions at infinity upstream are uniform, the magnetic field being $\mathfrak{B}_0 = \mathfrak{B}_0 \hat{\mathfrak{i}}_x$, and the number density of ions and electrons being N. Then, in a frame of reference moving with the wave, the motion appears steady and all quantities being functions only of x. We exclude from consideration those situations in which the particle trajectories are looped, so that all quantities are taken to be single-valued functions of x.

Reprint requests to Herrn B. K. Shivamoggi, Department of Theoretical Physics, Research School of Physical Sciences, The Australian National University, P. O. Box 4, Canberra A.C.T. 2600, Australien.

Let us nondimensionalise all quantities by using U, N, B_0, and a characteristic length

$$L = \frac{v_{\mathrm{A}}\, c \sqrt{m_{\mathrm{e}}\, m_{\mathrm{i}}}}{e\, B_0},$$

where m_{s} is the mass of species s (s refers to ions and electrons), e_{s} the electric charge ($|e_{\mathrm{s}}| = e$), c the velocity of light, and

$$v_{\mathrm{A}} \equiv \frac{B_0}{\sqrt{4\pi N (m_{\mathrm{i}} + m_{\mathrm{e}})}}.$$

If n_{s} and u_{s} denote the number density and the velocity component in the x-direction, the condition of mass-conservation requires

$$n_{\mathrm{s}}\, u_{\mathrm{s}} = 1 \tag{1}$$

with

$$x \Rightarrow -\infty:\ n_{\mathrm{s}} \Rightarrow 1\,, \quad u_{\mathrm{s}} \Rightarrow 1\,. \tag{2}$$

Let us now assume that the plasma is nearly electrical neutral (any small charge imbalance being capable of producing a finite electrostatic field, however) so that $n_{\mathrm{e}} \approx n_{\mathrm{i}} = n$. Equation (1) then gives

$$u_{\mathrm{e}} \approx u_{\mathrm{i}} = u\,. \tag{3}$$

Taking the electric field and the magnetic field to be

$$\left.\begin{aligned} \mathfrak{E} &= E\,\hat{\mathfrak{i}}_x, \\ \mathfrak{B} &= B_0\,\hat{\mathfrak{i}}_x + B_{1y}\,\hat{\mathfrak{i}}_y + B_{1z}\,\hat{\mathfrak{i}}_z, \end{aligned}\right\}$$

and the velocity of species s to be

$$v_{\mathrm{s}} = (u, v_{\mathrm{s}}, w_{\mathrm{s}})$$

and using the energy equation in the form

$$u \frac{\mathrm{d}P_{\mathrm{s}}}{\mathrm{d}x} + 3\,P_{\mathrm{s}} \frac{\mathrm{d}u}{\mathrm{d}x} = 0\,. \tag{4}$$

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P_s being the pressure, the equations of motion and Maxwell's equations give

$$(1 - V_{\mathrm{Ti}}^2)\frac{M}{\gamma} u \frac{\mathrm{d}u}{\mathrm{d}x} = E - w_i B_{1y} + v_i B_{1z}, \tag{5a}$$

$$(1 - V_{\mathrm{Te}}^2) M \gamma u \frac{\mathrm{d}u}{\mathrm{d}x} = -E + w_e B_{1y} - v_e B_{1z}, \tag{5b}$$

$$\frac{M}{\gamma} u \frac{\mathrm{d}v_i}{\mathrm{d}x} = -u B_{1z} + w_i, \tag{6a}$$

$$M \gamma u \frac{\mathrm{d}v_e}{\mathrm{d}x} = u B_{1z} - w_e, \tag{6b}$$

$$\frac{M}{\gamma} u \frac{\mathrm{d}w_i}{\mathrm{d}x} = u B_{1y} - v_i, \tag{7a}$$

$$M \gamma u \frac{\mathrm{d}w_e}{\mathrm{d}x} = -u B_{1y} + v_e, \tag{7b}$$

$$\frac{\mathrm{d}B_{1y}}{\mathrm{d}x} = \frac{M n}{\gamma + 1/\gamma}(w_i - w_e), \tag{8a}$$

$$\frac{\mathrm{d}B_{1z}}{\mathrm{d}x} = -\frac{M n}{\gamma + 1/\gamma}(v_i - v_e), \tag{8b}$$

where

$$\left.\begin{aligned} M &\equiv \frac{U}{v_A}, \quad \gamma^2 = \frac{m_e}{m_i}, \\ V_{\mathrm{Ts}} &= \frac{(3 P_s/m_s N)^{1/2}}{U}. \end{aligned}\right\}$$

P_s being the pressure of upstream infinity, and $(3P_s/m_s N)^{1/2}$ the thermal speed of the species s. In (8) we have neglected the displacement current, this being valid if the plasma is nonrelativistic and the wave-frequency in question is small compared with the plasma frequency (Ferraro and Plumpton [5]).

3. Solitary Waves in a Warm Plasma

From (7) and (8), and using the conditions at infinity, one obtains

$$B_{1y} = \frac{M^2}{\gamma + 1/\gamma}\left(\frac{v_i}{\gamma} + \gamma v_e\right), \tag{9a}$$

$$B_{1z} = \frac{M^2}{\gamma + 1/\gamma}\left(\frac{w_i}{\gamma} + \gamma w_e\right). \tag{9b}$$

Using (9), and eliminating E from eqn. (5), one obtains

$$\left[\gamma(1 - V_{\mathrm{Te}}^2) + \frac{1}{\gamma}(1 - V_{\mathrm{Ti}}^2)\right] u \frac{\mathrm{d}u}{\mathrm{d}x} = M(v_i w_e - v_e w_i). \tag{10}$$

Using (6a), (7a), (9) and (10) one obtains

$$\frac{M}{\gamma}\left(v_i \frac{\mathrm{d}v_i}{\mathrm{d}x} + w_i \frac{\mathrm{d}w_i}{\mathrm{d}x}\right) = w_i B_{1y} - v_i B_{1z} = \frac{M^2 \gamma}{\gamma + 1/\gamma}(w_i v_e - w_e v_i) = -M \gamma \alpha u (\mathrm{d}u/\mathrm{d}x)$$

so that

$$v_i^2 + w_i^2 = \alpha \gamma^2 (1 - u^2), \tag{11a}$$

where

$$\alpha \equiv \left\{\gamma(1 - V_{\mathrm{Te}}^2) + \frac{1}{\gamma}(1 - V_{\mathrm{Ti}}^2)\right\} \Big/ \left(\gamma + \frac{1}{\gamma}\right). \tag{12}$$

Similarly,

$$v_e^2 + w_e^2 = (\alpha/\gamma^2)(1 - u^2). \tag{11b}$$

Note from (11) that the existence of a realistic solution in a warm plasma requires $u < 1$ (as in a cold plasma).

Using (1), (3) and (8), Eq. (5) gives

$$\alpha M^2 \frac{\mathrm{d}u}{\mathrm{d}x} + B_{1y}\frac{\mathrm{d}B_{1y}}{\mathrm{d}x} + B_{1z}\frac{\mathrm{d}B_{1z}}{\mathrm{d}x} = 0$$

or on using (2), this gives

$$\alpha M^2 u + \tfrac{1}{2}(B_{1y}^2 + B_{1z}^2) = \alpha M^2. \tag{13}$$

Using (9) and (11), (13) gives

$$(w_i w_e + v_i v_e) = \frac{1 - u}{\alpha}\left(\frac{\gamma + 1/\gamma}{M^2}\right) - \alpha(1 - u^2). \tag{14}$$

Using the identity

$$(v_i v_e + w_i w_e)^2 = (v_i^2 + w_i^2)(v_e^2 + w_e^2) - (v_i w_e - v_e w_i)^2 \tag{15}$$

and (10) and (11), (14) gives

$$\alpha^2\left(u \frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 = (1 - u^2)\left[2(1 + u) - \frac{(\gamma + 1/\gamma)^2 \alpha^2}{M^2}\right]. \tag{16}$$

Upon introducing a new independent variable t, according to

$$\frac{1}{\alpha}\frac{\mathrm{d}x}{\mathrm{d}t} = u. \tag{17}$$

Eq. (16) gives a solitary wave

$$u = 1 - 2\lambda^2 \operatorname{sec} h^2 \lambda t, \tag{18}$$

where

$$\lambda^2 = 1 - \frac{(\gamma + 1/\gamma)^2 \alpha^2}{4 M^2}. \tag{19}$$

One observes from (18), (19) and (13) that the effect of thermal motion of ions and electrons is to reduce the amplitude but enhance the velocity of propagation of the solitary wave. An identical effect of a finite ion-temperature is observed for ion-acoustic solitary waves (Tappert, [6]; Tagare, [7]; Shivamoggi, [8]).

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